

EXTREME EIGENVALUES OF AN INTEGRAL OPERATOR

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ABSTRACT. We study the family of compact operators $B_\alpha = VA_\alpha V$, $\alpha > 0$ in $L^2(\mathbb{R}^d)$, $d \geq 1$, where A_α is the pseudo-differential operator with symbol $a_\alpha(\xi) = a(\alpha\xi)$, and both functions a and V are real-valued and decay at infinity. We assume that a and V attain their maximal values $A_0 > 0$, $V_0 > 0$ only at $\xi = 0$ and $\mathbf{x} = 0$. We also assume that

$$a(\xi) = A_0 - \Psi_\gamma(\xi) + o(|\xi|^\gamma), \quad |\xi| \rightarrow 0,$$

$$V(\mathbf{x}) = V_0 - \Phi_\beta(\mathbf{x}) + o(|\mathbf{x}|^\beta), \quad |\mathbf{x}| \rightarrow 0,$$

with some functions $\Psi_\gamma(\xi) > 0$, $\xi \neq 0$ and $\Phi_\beta(\mathbf{x}) > 0$, $\mathbf{x} \neq 0$ that are homogeneous of degree $\gamma > 0$ and $\beta > 0$ respectively. The main result is the following asymptotic formula for the eigenvalues $\lambda_\alpha^{(n)}$ of the operator B_α (arranged in descending order counting multiplicity) for fixed n and $\alpha \rightarrow 0$:

$$\lambda_\alpha^{(n)} = A_0 V_0^2 - \mu^{(n)} \alpha^\sigma + o(\alpha^\sigma), \quad \alpha \rightarrow 0,$$

where $\sigma^{-1} = \gamma^{-1} + \beta^{-1}$, and $\mu^{(n)}$ are the eigenvalues (arranged in ascending order counting multiplicity) of the model operator T with symbol $V_0^2 \Psi_\gamma(\xi) + 2A_0 V_0 \Phi_\beta(\mathbf{x})$.

1. INTRODUCTION AND MAIN RESULT

Let $a = a(\xi)$, $\xi \in \mathbb{R}^d$, $V = V(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, $d \geq 1$, be bounded real-valued functions such that $a(\xi) \rightarrow 0$, $V(\mathbf{x}) \rightarrow 0$ as $|\xi| \rightarrow \infty$, $|\mathbf{x}| \rightarrow \infty$. Consider the self-adjoint operator on $L^2(\mathbb{R}^d)$ defined by

$$B_\alpha = V \mathcal{F}^* a_\alpha \mathcal{F} V, \quad a_\alpha(\xi) = a(\alpha\xi), \quad \alpha > 0,$$

where \mathcal{F} is the unitary Fourier transform

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int e^{-i\xi \cdot \mathbf{x}} u(\mathbf{x}) d\mathbf{x}.$$

Here and further on the integral without indication of the domain means integration over the entire space \mathbb{R}^d . The operator $\mathcal{F}^* a_\alpha \mathcal{F}$ is also described as a pseudo-differential operator with symbol a_α . This description however is not helpful for us as we do not use calculus of pseudo-differential operators. It is clear that B_α is compact for all $\alpha > 0$. We are interested in the asymptotics of the extreme top eigenvalues of the operator B_α as $\alpha \rightarrow 0$. More precisely, denote by $\lambda_\alpha^{(1)}, \lambda_\alpha^{(2)}, \dots$ the eigenvalues of B_α arranged in

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descending order counting multiplicity. The associated normalized pair-wise orthogonal eigenfunctions are denoted by $\psi_\alpha^{(1)}, \psi_\alpha^{(2)}, \dots$. We study the asymptotics of $\lambda_\alpha^{(n)}$ as $\alpha \rightarrow 0$ for a fixed n . This problem has been addressed in the literature in different contexts under different conditions on the functions a and V . For example, if a and V are indicator functions of bounded intervals in \mathbb{R} , the behaviour of the eigenvalues was studied by D. Slepian and H.O. Pollak in [7]. For $d \geq 2$ this problem was analyzed by D. Slepian in [8] with a, V being indicator functions of balls. In both cases (one- and multi-dimensional) the eigenvalues $\lambda_\alpha^{(n)}$ are exponentially close to 1 as $\alpha \rightarrow 0$.

In [9] H. Widom considered the function V which was the indicator of an interval I , and symbol $a = a(\xi)$, $\xi \in \mathbb{R}$, having one global maximum at $\xi = 0$, and satisfying the condition

$$(1.1) \quad a(\xi) = A_0 - \Psi|\xi|^\gamma + o(|\xi|^\gamma), \quad |\xi| \rightarrow 0,$$

with $A_0 = a(0) = \max a(\xi) > 0$, and some $\Psi > 0$, $\gamma > 0$. It was proved that

$$(1.2) \quad \lambda_\alpha^{(n)} = A_0 - \alpha^\gamma \Psi \mu^{(n)} + o(\alpha^\gamma), \quad \alpha \rightarrow 0,$$

where $\mu^{(n)}$, $n = 1, 2, \dots$ are eigenvalues of the fractional Dirichlet Laplacian $(-\Delta)^{\frac{\gamma}{2}}$ on I , arranged in ascending order counting multiplicity. A multi-dimensional analogue of this result was obtained by H. Widom in [10]. We omit its formulation for the sake of brevity. A result of the type (1.1) also holds if V is not assumed to be a simple indicator function, but attains its (positive) maximum on a set of positive measure, see [5].

For applications to transport problems (see [2] and [3]) it is also useful to investigate the case where both functions a and V have unique power-like maxima. This is exactly the case that we study in the present paper. The precise conditions on a and V are described below. By C, c (with or without indices) we denote various positive constants whose precise value is of no importance.

Condition 1.1. (1) a and V are real-valued L^∞ -functions such that $a(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$, and $V(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$.

(2) The functions a and V attain their global maxima only at $\xi = 0$ and $\mathbf{x} = 0$ respectively:

$$A_0 := \max a(\xi) > 0, \quad V_0 := \max V(\mathbf{x}) > 0.$$

The function V satisfies the condition $-V_0 + c \leq V(\mathbf{x}) \leq V_0$, \mathbf{x} a.e., with a positive constant c .

(3) Let $\Phi_\beta, \Psi_\gamma \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be some real-valued functions, homogeneous of degree $\beta > 0$ and $\gamma > 0$ respectively, positive at $\mathbf{x} \neq 0$. The functions V and a satisfy the properties

$$(1.3) \quad V(\mathbf{x}) = V_0 - \Phi_\beta(\mathbf{x}) + o(|\mathbf{x}|^\beta), \quad |\mathbf{x}| \rightarrow 0,$$

and

$$(1.4) \quad a(\xi) = A_0 - \Psi_\gamma(\xi) + o(|\xi|^\gamma), \quad |\xi| \rightarrow 0.$$

The results are described with the help of the following model pseudo-differential operator T defined formally by its symbol

$$(1.5) \quad t(\mathbf{x}, \boldsymbol{\xi}) = V_0^2 \Psi_\gamma(\boldsymbol{\xi}) + 2A_0 V_0 \Phi_\beta(\mathbf{x}).$$

The operator T is essentially self-adjoint on $C_0^\infty(\mathbb{R})$, and has a purely discrete spectrum (see e.g. [6, Theorems 26.2, 26.3]). The same operator can be also defined (see [1, p. 229, Theorem 1]) as the unique self-adjoint operator associated with the quadratic form

$$(1.6) \quad T[u, v] = V_0^2 \int \Psi_\gamma(\boldsymbol{\xi}) \hat{u}(\boldsymbol{\xi}) \overline{\hat{v}(\boldsymbol{\xi})} d\boldsymbol{\xi} + 2A_0 V_0 \int \Phi_\beta(\mathbf{x}) u(\mathbf{x}) \overline{v(\mathbf{x})} d\mathbf{x},$$

which is closed on $D[T] = H^{\frac{\gamma}{2}}(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^\beta)$. We use the notation $T[u] = T[u, u]$. Recall that in view of the polarization identity, the form $T[w], w \in D[T]$, determines $T[u, v]$ for all $u, v \in D[T]$. Denote by $\mu^{(n)} > 0$, $n = 1, 2, \dots$ the eigenvalues of T arranged in ascending order counting multiplicity, and by $\phi^{(n)}$ – an orthonormal basis of corresponding normalized eigenfunctions.

Let σ be the number found from the equation

$$\frac{1}{\sigma} = \frac{1}{\beta} + \frac{1}{\gamma}.$$

The next theorem constitutes the main result of the paper.

Theorem 1.2. *Suppose that the functions a and V satisfy Condition 1.1. Then for any $n = 1, 2, \dots$, the asymptotics hold:*

$$(1.7) \quad \lim_{\alpha \rightarrow 0} \alpha^{-\sigma} (A_0 V_0^2 - \lambda_\alpha^{(n)}) = \mu^{(n)}.$$

Let us make a few remarks.

Note that formally, the asymptotics (1.7) imply (1.2) if one takes $d = 1$ and $\beta = \infty$.

Observe also that a model operator of the form (1.5) was featured in [4] where the norm of a special self-adjoint integral operator with properties similar to B_α , was studied.

One could also examine the case when one or both of the functions a, V attain their respective maximum values at several points, and have there the asymptotics of the type (1.3) and (1.4). The author believes that this problem can be tackled by standard methods via decoupling distinct maximal points, thereby reducing the issue to the case of a single maximum.

Conceptually, the proof of Theorem 1.2 follows the paper [9], but the technical details are quite different: for instance, the model operator T replaces the fractional Laplacian used in [9].

2. PRELIMINARY ESTIMATES. LOWER BOUNDS FOR THE TOP EIGENVALUES

Throughout the paper we assume that Condition 1.1 is satisfied. Without loss of generality we may assume that $A_0 = V_0 = 1$.

Using the unitary scaling transformation reduce the studied operator to the operator

$$B_\alpha = W_\alpha \text{Op}(b_\alpha) W_\alpha,$$

where W_α, a_α are defined in the following way:

$$W_\alpha(\mathbf{x}) = V(\alpha^{\frac{\gamma}{\gamma+\beta}} \mathbf{x}), \quad b_\alpha(\boldsymbol{\xi}) = a(\alpha^{\frac{\beta}{\gamma+\beta}} \boldsymbol{\xi}).$$

Note that slightly abusing the notation we use for the unitarily equivalent operator the same symbol B_α . This will not cause any confusion. For thus defined functions W_α and b_α the conditions (1.3) and (1.4) imply that

$$(2.1) \quad \lim_{\alpha \rightarrow 0} \alpha^{-\sigma} (1 - b_\alpha(\boldsymbol{\xi})) = \Psi_\gamma(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,$$

and

$$(2.2) \quad \lim_{\alpha \rightarrow 0} \alpha^{-\sigma} (1 - W_\alpha(\mathbf{x})^2) = 2\Phi_\beta(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Both convergences are uniform in \mathbf{x} and $\boldsymbol{\xi}$ varying over compact sets.

Here is another useful property of the family W_α :

Lemma 2.1. *For any $u \in D[T]$, we have*

$$(2.3) \quad \alpha^{-\sigma} \int |W_\alpha(\mathbf{x}) - 1|^2 |u(\mathbf{x})|^2 d\mathbf{x} \rightarrow 0, \quad \alpha \rightarrow 0.$$

Proof. The function $W_\alpha - 1$ is bounded uniformly in \mathbf{x} and α , so that

$$|W_\alpha(\mathbf{x}) - 1|^2 \leq C |W_\alpha(\mathbf{x}) - 1|, \quad \mathbf{x} \in \mathbb{R}^d.$$

On the other hand,

$$|W_\alpha(\mathbf{x}) - 1| \leq C \alpha^\sigma |\mathbf{x}|^\beta, \quad \mathbf{x} \in \mathbb{R}^d.$$

Therefore, for any $R > 0$, we can estimate as follows:

$$\begin{aligned} & \alpha^{-\sigma} \int |W_\alpha(\mathbf{x}) - 1|^2 |u(\mathbf{x})|^2 d\boldsymbol{\xi} \\ & \leq \alpha^{-\sigma} \int_{|\mathbf{x}| < R} |W_\alpha(\mathbf{x}) - 1|^2 |u(\mathbf{x})|^2 d\mathbf{x} + C \alpha^{-\sigma} \int_{|\mathbf{x}| \geq R} |W_\alpha(\mathbf{x}) - 1| |u(\mathbf{x})|^2 d\mathbf{x} \\ & \leq C \alpha^\sigma \int_{|\mathbf{x}| < R} |\mathbf{x}|^{2\beta} |u(\mathbf{x})|^2 d\mathbf{x} + C \int_{|\mathbf{x}| > R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x} \\ & \leq C \alpha^\sigma R^\beta \int_{|\mathbf{x}| < R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\boldsymbol{\xi} + C \int_{|\mathbf{x}| > R} |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

Both integrals on the right-hand side are finite, since $u \in D[T]$, and the second one tends to zero as $R \rightarrow \infty$. Thus, passing first to the limit $\alpha \rightarrow 0$, and then taking $R \rightarrow \infty$, we conclude that the right-hand side tends to zero as $\alpha \rightarrow 0$, as claimed. \square

Now we show that in some suitable sense the operator B_α can be approximated by the operator $I - \alpha^\sigma T$ as $\alpha \rightarrow 0$. Define the form

$$(2.4) \quad R_\alpha[u] = (B_\alpha u, u) - \|u\|^2 + \alpha^\sigma T[u],$$

which is closed on the domain $D[T]$, and two more forms

$$(2.5) \quad K_\alpha[u, v] = \alpha^{-\sigma} \int (1 - b_\alpha(\boldsymbol{\xi})) \hat{u}(\boldsymbol{\xi}) \overline{\hat{v}(\boldsymbol{\xi})} d\boldsymbol{\xi},$$

$$(2.6) \quad S_\alpha[u, v] = \alpha^{-\sigma} \int (1 - W_\alpha(\mathbf{x})^2) u(\mathbf{x}) \overline{v(\mathbf{x})} d\mathbf{x},$$

that are defined for all $u, v \in L^2(\mathbb{R}^d)$. It is easily checked that with $w_\alpha = W_\alpha u$, $y_\alpha = W_\alpha v$, we have

$$(2.7) \quad \alpha^{-\sigma} ((u, v) - (B_\alpha u, v)) = K_\alpha[w_\alpha, y_\alpha] + S_\alpha[u, v],$$

and

$$(2.8) \quad R_\alpha[u, v] = \alpha^\sigma (T[u, v] - K_\alpha[w_\alpha, y_\alpha] - S_\alpha[u, v]).$$

Note that $K_\alpha[u] \geq 0$ and $S_\alpha[u] \geq 0$ for all $\alpha > 0$. Also, due to (2.1) and (2.2), for any $u \in D[T]$ we have

$$(2.9) \quad K_\alpha[u] \leq C \int |\boldsymbol{\xi}|^\gamma |\hat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}, \quad S_\alpha[u] \leq C \int |\mathbf{x}|^\beta |u(\mathbf{x})|^2 d\mathbf{x},$$

with a constant C independent of u . Moreover, for any $u \in D[T]$ we also have

$$(2.10) \quad \lim_{\alpha \rightarrow 0} K_\alpha[u] = \int \Psi_\gamma(\boldsymbol{\xi}) |\hat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

and

$$(2.11) \quad \lim_{\alpha \rightarrow 0} S_\alpha[u] = 2 \int \Phi_\beta(\mathbf{x}) |u(\mathbf{x})|^2 d\mathbf{x},$$

by the Dominated Convergence Theorem.

Lemma 2.2. *For any $u \in D[T]$ and $w_\alpha = W_\alpha u$, we have*

$$(2.12) \quad K_\alpha[w_\alpha - u] \rightarrow 0, \quad \alpha \rightarrow 0.$$

Also, for any $u, v \in D[T]$ we have

$$(2.13) \quad \alpha^{-\sigma} |R_\alpha[u, v]| \rightarrow 0, \quad \alpha \rightarrow 0.$$

Proof. Proof of (2.12). Estimate:

$$\begin{aligned} K_\alpha[w_\alpha - u] &\leq C \alpha^{-\sigma} \int |w_\alpha(\mathbf{x}) - u(\mathbf{x})|^2 d\mathbf{x} \\ &= C \alpha^{-\sigma} \int (1 - W_\alpha(\mathbf{x}))^2 |u(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

Here we used the fact that $0 \leq 1 - b_\alpha \leq C$ with some constant C . The right-hand side tends to zero by (2.3).

It suffices to prove (2.13) for $u = v$. Consider separately the terms in the representation (2.8). Write:

$$K_\alpha[w_\alpha] = K_\alpha[u] + 2 \operatorname{Re} K_\alpha[u, w_\alpha - u] + K_\alpha[w_\alpha - u].$$

The last term tends to zero by (2.12). Now estimate the second term:

$$|K_\alpha[u, w_\alpha - u]|^2 \leq K_\alpha[u] K_\alpha[w_\alpha - u].$$

In view of (2.9), the first factor is uniformly bounded, and the second one tends to zero. Thus

$$K_\alpha[w_\alpha] - K_\alpha[u] \rightarrow 0, \quad \alpha \rightarrow 0.$$

Together with (2.10) and (2.11) this implies that

$$\lim_{\alpha \rightarrow 0} (K_\alpha[w_\alpha] + S_\alpha[u]) = T[u],$$

see (1.6). Due to (2.8) this implies (2.13). \square

The lower bound for the eigenvalues $\lambda_\alpha^{(n)}$, i.e. the upper bound for the left-hand side of (1.7), is rather straightforward.

Lemma 2.3. *For all $n = 1, 2, \dots$, we have*

$$(2.14) \quad \limsup_{\alpha \rightarrow \infty} \alpha^{-\sigma} (1 - \lambda_\alpha^{(n)}) \leq \mu^{(n)}.$$

Proof. Let $\mathcal{K}_n \subset L^2(\mathbb{R}^d)$, $n \geq 1$, be the span of the eigenfunctions $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(n)}$, so $\dim \mathcal{K}_n = n$. By the max-min principle (see e.g. [1, p. 212, Theorem 5]),

$$\lambda_\alpha^{(n)} \geq \min(B_\alpha u, u),$$

where the minimum is taken over all functions $u \in \mathcal{K}_n$ such that $\|u\| = 1$. Thus by definition (2.4)

$$\lambda_\alpha^{(n)} \geq 1 - \alpha^\sigma \max_{u \in \mathcal{K}_n, \|u\|=1} T[u] - n \max_{1 \leq j, k \leq n} |R_\alpha[\phi^{(j)}, \phi^{(k)}]|.$$

Since $\{\phi^{(j)}\}$ are eigenfunctions of T ,

$$\max_{u \in \mathcal{K}_n, \|u\|=1} T[u] = \mu^{(n)},$$

and the required result now follows from (2.13). \square

Now we can establish the uniform localization of the eigenfunctions $\psi_\alpha^{(n)}$, $n = 1, 2, \dots$. Denote

$$\theta_\alpha^{(n)}(\mathbf{x}) = W_\alpha(\mathbf{x}) \psi_\alpha^{(n)}(\mathbf{x}).$$

Lemma 2.4. *For all $n = 1, 2, \dots$, the forms $K_\alpha[\theta_\alpha^{(n)}]$ and $S_\alpha[\psi_\alpha^{(n)}]$ are bounded uniformly in α :*

$$(2.15) \quad \limsup_{\alpha \rightarrow 0} (K_\alpha[\theta_\alpha^{(n)}] + S_\alpha[\psi_\alpha^{(n)}]) \leq \mu^{(n)},$$

and

$$(2.16) \quad \|\theta_\alpha^{(n)} - \psi_\alpha^{(n)}\| \rightarrow 0, \quad \alpha \rightarrow 0.$$

Moreover, for all $R > 0$ we have

$$(2.17) \quad \liminf_{\alpha \rightarrow 0} \|\widehat{\psi_\alpha^{(n)}} \chi_R\|^2 \geq 1 - C\mu^{(n)}R^{-\gamma},$$

and

$$(2.18) \quad \liminf_{\alpha \rightarrow 0} \|\psi_\alpha^{(n)} \chi_R\|^2 \geq 1 - C\mu^{(n)}R^{-\beta}.$$

with some constant C , independent of n and R .

Proof. We drop the superscript “ n ” for brevity. According to (2.7),

$$\alpha^{-\sigma}(1 - \lambda_\alpha) = K_\alpha[\theta_\alpha] + S_\alpha[\psi_\alpha].$$

Now (2.15) follows from (2.14). Now write

$$\|\theta_\alpha - \psi_\alpha\|^2 = \int (1 - W_\alpha(\mathbf{x}))^2 |\psi_\alpha(\mathbf{x})|^2 d\mathbf{x}.$$

The straightforward estimate

$$\frac{1}{2}(1 - W_\alpha)^2 \leq 1 - W_\alpha = \frac{1 - W_\alpha^2}{1 + W_\alpha} \leq C(1 - W_\alpha^2),$$

by the definition (2.6), implies that

$$\|\theta_\alpha - \psi_\alpha\|^2 \leq C\alpha^\sigma S_\alpha[\psi_\alpha],$$

which leads to the convergence $\|\theta_\alpha - \psi_\alpha\| \rightarrow 0$, $\alpha \rightarrow 0$, in view of (2.15).

Proof of (2.17). By Condition 1.1(2), the point $\boldsymbol{\xi} = \mathbf{0}$ is the global maximum of $b_\alpha(\boldsymbol{\xi})$, so in view of (1.4), for all $|\boldsymbol{\xi}| > R$, $R > 0$ and all sufficiently small α we have

$$b_\alpha(\boldsymbol{\xi}) = a(\alpha^{\frac{\beta}{\gamma+\beta}} \boldsymbol{\xi}) \leq 1 - CR^\gamma \alpha^\sigma,$$

with some constant C . Thus $\alpha^{-\sigma}(1 - b_\alpha(\boldsymbol{\xi})) \geq CR^\gamma$, and hence

$$K_\alpha[\theta_\alpha] \geq CR^\gamma \int_{|\boldsymbol{\xi}| > R} |\hat{\theta}_\alpha(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

so that, by (2.15), $\|\hat{\theta}_\alpha(1 - \chi_R)\|^2 \leq C\mu R^{-\gamma}$. Together with (2.16) this leads to (2.17).

Proof of (2.18) is similar. By Condition 1.1(2) and by (1.3), for all $|\mathbf{x}| > R$, $R > 0$, we have $|W_\alpha(\mathbf{x})|^2 \leq 1 - CR^\beta \alpha^\sigma$, and hence

$$S_\alpha[\psi_\alpha] \geq CR^\beta \int_{|\mathbf{x}| > R} |\psi_\alpha(\mathbf{x})|^2 d\mathbf{x},$$

so that by (2.15) again, $\|\psi_\alpha(1 - \chi_R)\|^2 \leq C\mu R^{-\beta}$. This leads to (2.18). \square

With the help of Lemma 2.4, in the proof of Theorem 1.2 we show that any weakly convergent sequence of the eigenfunctions $\psi_\alpha^{(n)}$ in fact converges in norm. For this we rely on the following result:

Proposition 2.5. (See [4, Lemma 12]) *Let $f_j \in \mathbf{L}^2(\mathbb{R}^d)$ be a sequence such that $\|f_j\| \leq C$ uniformly in $j = 1, 2, \dots$, and $f_j(\mathbf{x}) = 0$ for all $|\mathbf{x}| \geq \rho > 0$ and all $j = 1, 2, \dots$. Suppose that f_j converges weakly to $f \in \mathbf{L}^2(\mathbb{R}^d)$ as $j \rightarrow \infty$, and that for some constant $A > 0$, and all $R \geq R_0 > 0$,*

$$(2.19) \quad \liminf_{j \rightarrow \infty} \|\hat{f}_j \chi_R\| \geq A - CR^{-\varkappa}, \quad \varkappa > 0,$$

with some constant C independent of j, R . Then $\|f\| \geq A$.

3. PROOF OF THEOREM 1.2

As before, we assume that a and V satisfy Condition 1.1, and that $A_0 = V_0 = 1$.

The next lemma is the last step towards the proof of Theorem 1.2.

Lemma 3.1. *Suppose that for some sequence $\alpha_k > 0$, convergent to zero as $k \rightarrow \infty$, the sequence of eigenfunctions $\psi_{\alpha_k}^{(n)}$ converges weakly to $\psi^{(n)}$. Then*

- (1) *The sequence $\psi_{\alpha_k}^{(n)}$ converges to $\psi^{(n)}$ in norm as $k \rightarrow \infty$,*
- (2) *The norm limit $\psi^{(n)}$ belongs to $D[T]$, and*

$$(3.1) \quad \lim_{k \rightarrow \infty} \alpha_k^{-\sigma} ((\psi_{\alpha_k}^{(n)}, g) - B_{\alpha_k}[\psi_{\alpha_k}^{(n)}, g]) = T[\psi^{(n)}, g],$$

for any $g \in D[T]$.

Proof. As before, we omit the superscript “ n ”. Also for brevity we write α instead of α_k .

Proof of (1). Due to the formula

$$\|\psi - \psi_\alpha\|^2 = 1 + \|\psi\|^2 - 2\operatorname{Re}(\psi_\alpha, \psi) \rightarrow 1 - \|\psi\|^2, \quad \alpha \rightarrow 0,$$

it suffices to show that $\|\psi\| = 1$.

For a number $\rho > 0$ denote $w_{\alpha, \rho} = \psi_\alpha \chi_\rho$, $y_{\alpha, \rho} = \psi_\alpha(1 - \chi_\rho)$. Thus, by (2.17) and (2.18),

$$\|\widehat{w_{\alpha, \rho} \chi_R}\| \geq \|\widehat{\psi_\alpha \chi_R}\| - \|y_{\alpha, \rho}\| \geq 1 - C\mu R^{-\gamma} - C(\mu\rho^{-\beta})^{\frac{1}{2}}.$$

Since $\psi_\alpha \rightarrow \psi$ weakly, then for any $\rho > 0$ the family $w_{\alpha,\rho}$ converges to $\psi\chi_\rho$ weakly. Using Proposition 2.5 for the sequence $w_{\alpha,\rho}$ we conclude that

$$\|\psi\chi_\rho\| \geq 1 - C(\mu\rho^{-\beta})^{\frac{1}{2}}.$$

Since ρ is arbitrary, this means that $\|\psi\| = 1$, which implies the norm convergence $\psi_\alpha \rightarrow \psi$, $\alpha \rightarrow 0$, as claimed.

Proof of (2). By Part (1) above, and by (2.16), we have

$$\|\hat{\theta}_\alpha - \hat{\psi}\| \leq \|\theta_\alpha - \psi_\alpha\| + \|\psi_\alpha - \psi\| \rightarrow 0, \alpha \rightarrow 0.$$

Thus for a subsequence $\hat{\theta}_\alpha$, there is a pointwise convergence $\hat{\theta}_\alpha \rightarrow \hat{\psi}$, $\alpha \rightarrow 0$. By (2.1), the integrand in $K_\alpha[\theta_\alpha]$ converges pointwise to $\Psi_\gamma(\xi)|\hat{\psi}(\xi)|^2$. By (2.15), $K_\alpha[\theta_\alpha]$ is uniformly bounded, so by Fatou's Lemma, $|\xi|^{\gamma/2}\hat{\psi} \in L^2(\mathbb{R}^d)$.

By (2.2), the integrand in $S_\alpha[\psi_\alpha]$ converges pointwise to $2\Phi_\beta(\mathbf{x})|\psi(\mathbf{x})|^2$. By (2.15), $S_\alpha[\psi_\alpha]$ is uniformly bounded, so by Fatou's Lemma again, $|\mathbf{x}|^{\beta/2}\psi \in L^2(\mathbb{R}^d)$. Together with the previously obtained property $|\xi|^{\gamma/2}\hat{\psi} \in L^2(\mathbb{R}^d)$, this means that $\psi \in D[T]$.

Proof of (3.1) is similar to that of (2.13), but is somewhat more complicated since it involves functions ψ_α depending on the parameter α . By (2.7),

$$\alpha^{-\sigma}((\psi_\alpha, g) - B_\alpha[\psi_\alpha, g]) = K_\alpha[\theta_\alpha, y_\alpha] + S_\alpha[\psi_\alpha, g],$$

where $y_\alpha = W_\alpha g$. We prove that

$$(3.2) \quad \lim_{\alpha \rightarrow 0} K_\alpha[\theta_\alpha, y_\alpha] = \int \Psi_\gamma(\xi)\hat{\psi}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

and

$$(3.3) \quad \lim_{\alpha \rightarrow 0} S_\alpha[\psi_\alpha, g] = 2 \int \Phi_\beta(\mathbf{x})\psi(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}.$$

Estimate:

$$|K_\alpha[\theta_\alpha, y_\alpha] - K_\alpha[\theta_\alpha, g]|^2 \leq K_\alpha[\theta_\alpha]K_\alpha[y_\alpha - g].$$

The first factor is bounded uniformly in α by (2.15), and the second one tends to zero due to (2.12). This shows that

$$(3.4) \quad K_\alpha[\theta_\alpha, y_\alpha] - K_\alpha[\theta_\alpha, g] \rightarrow 0, \alpha \rightarrow 0.$$

Because of this property, and because of (2.9), in the proof of (3.2) we may assume that \hat{g} is compactly supported, i.e. $\hat{g}(\xi) = 0$ for all $|\xi| > R$ with some $R > 0$. The convergence (2.1) is uniform in $\xi : |\xi| \leq R$ for any R . At the same time, as shown earlier, $\|\hat{\theta}_\alpha - \hat{\psi}\| \rightarrow 0, \alpha \rightarrow 0$, so that

$$K_\alpha[\theta_\alpha, g] \rightarrow \int \Psi_\gamma(\xi)\hat{\psi}(\xi)\overline{\hat{g}(\xi)}d\xi, \alpha \rightarrow 0.$$

Together with (3.4) this gives (3.2).

Proof of (3.3) is simpler. Because of (2.9), we may assume that g is compactly supported. The convergence (2.2) is uniform in $\mathbf{x} : |\mathbf{x}| \leq R$ for any $R > 0$. Using the property $\|\psi_\alpha - \psi\| \rightarrow 0, \alpha \rightarrow 0$, established in Part 1, we obtain

$$S_\alpha[\psi_\alpha, g] \rightarrow \int 2\Phi_\beta(\mathbf{x})\psi(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}, \quad \alpha \rightarrow 0,$$

so that (3.3) holds.

Put together (3.2) and (3.3) to conclude that

$$\alpha^{-\sigma}((\psi_\alpha, g) - B_\alpha[\psi_\alpha, g]) \rightarrow T[\psi, g], \quad \alpha \rightarrow 0,$$

as required. \square

Proof of Theorem 1.2. The proof essentially follows the plan of [9]. It suffices to show that for any sequence $\alpha_k \rightarrow 0, k \rightarrow \infty$, one can find a subsequence $\alpha_{k_l} \rightarrow 0, l \rightarrow \infty$, such that

$$(3.5) \quad \lim_{l \rightarrow \infty} \alpha_{k_l}^{-\sigma}(1 - \lambda_{\alpha_{k_l}}^{(n)}) = \mu^{(n)}.$$

Since $\|\psi_{\alpha_k}^{(n)}\| = 1$, one can extract a subsequence $\alpha_{k_l} \rightarrow 0$ such that $\psi_{\alpha_{k_l}}^{(n)}$ converges weakly as $l \rightarrow \infty$. By Lemma 3.1 $\psi_{\alpha_{k_l}}^{(n)}$ converges in norm as $l \rightarrow \infty$. Denote by $\psi^{(n)}$ its limit, so $\|\psi^{(n)}\| = 1$. Further for simplicity we write $\psi_\alpha^{(n)}$ and $\lambda_\alpha^{(n)}$ instead of $\psi_{\alpha_{k_l}}^{(n)}$ and $\lambda_{\alpha_{k_l}}^{(n)}$. As $\psi_\alpha^{(n)}, n = 1, 2, \dots$, are pair-wise orthogonal, so are their limits $\psi^{(n)}, n = 1, 2, \dots$.

Fix a number $n = 1, 2, \dots$. For an arbitrary function $f \in D[T]$ write

$$\alpha^{-\sigma}(1 - \lambda_\alpha^{(n)})(\psi_\alpha^{(n)}, f) = \alpha^{-\sigma}((\psi_\alpha^{(n)}, f) - B_\alpha[\psi_\alpha^{(n)}, f]).$$

Suppose that f is such that $(\psi^{(n)}, f) \neq 0$. Then, in view of (3.1),

$$\lim_{\alpha \rightarrow 0} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) = \frac{T[\psi^{(n)}, f]}{(\psi^{(n)}, f)}.$$

Let $f = \phi^{(j)}$, where $\phi^{(j)}$ is chosen in such a way that $(\phi^{(j)}, \psi^{(n)}) \neq 0$. This is possible due to the completeness of the family $\phi^{(k)}, k = 1, 2, \dots$. Thus

$$\lim_{\alpha \rightarrow 0} \alpha^{-\sigma}(1 - \lambda_\alpha^{(n)}) = \mu^{(j)}.$$

By the uniqueness of the above limit, $(\psi^{(j)}, \phi^{(s)}) = 0$ for all s 's such that $\mu^{(s)} \neq \mu^{(j)}$. Thus, by completeness of the system $\{\phi^{(k)}\}$, the function $\psi^{(n)}$ is an eigenfunction of T with the eigenvalue $\mu^{(j)}$, i.e. $T[\psi^{(n)}] = \mu^{(j)}\psi^{(n)}$. As $\psi_\alpha^{(k)}, k = 0, 1, \dots, n$, are pair-wise orthogonal, so are their limits $\psi^{(k)}, k = 0, 1, \dots, n$.

Further proof is by induction. Let $n = 1$, so that by (2.14), $\mu^{(j)} \leq \mu^{(1)}$, and hence $j = 1$, and $\psi^{(1)}$ is the eigenfunction of T with eigenvalue $\mu^{(1)}$. Suppose that for some n , the collection $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(n-1)}$ are eigenfunctions of T with eigenvalues $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n-1)}$. Since $\psi^{(n)}$ is orthogonal to each $\psi^{(k)}, k = 1, 2, \dots, n-1$, by the standard min-max (or,

more precisely, max-min) principle for operators semi-bounded from below, we have $T[\psi^{(n)}] \geq \mu^{(n)}$, which means that $\mu^{(j)} \geq \mu^{(n)}$. On the other hand, by (2.14),

$$\lim_{\alpha \rightarrow 0} \alpha^{-\sigma} (1 - \lambda_{\alpha}^{(n)}) \leq \mu^{(n)},$$

and hence $\mu^{(j)} \leq \mu^{(n)}$. Therefore $\mu^{(j)} = \mu^{(n)}$, and $\psi^{(n)}$ is the eigenfunction of T with eigenvalue $\mu^{(n)}$. By induction, the formula (3.5) is proved for all n , which entails (1.7), and hence proves Theorem 1.2. \square

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